Desarrollos de Taylor

① $f(z) = e^z$ en torno a $z_0 = i$

Sabemos: $e^{\omega} = 1 + \omega + \dfrac{\omega^2}{2!} + \dfrac{\omega^3}{3!} + \dots = \sum_{k=0}^{\infty} \dfrac{\omega^k}{k!}$ para todo $\omega \in \mathbb{C}$

Si $\omega = z - i$:

$$e^{z-i} = 1 + (z-i) + \frac{(z-i)^2}{2!} + \frac{(z-i)^3}{3!} + \dots = \sum \frac{(z-i)^k}{k!} \quad \text{para todo } z$$

$$e^z e^{-i} = \dots\dots$$

$$e^z = e^i + e^i(z-i) + e^i\frac{(z-i)^2}{2!} + e^i\frac{(z-i)^3}{3!} + \dots = \sum \frac{e^i}{k!}(z-i)^k$$

Coef de la serie: $a_k = \dfrac{e^i}{k!}$

Radio de convergencia: $R = \infty$.

② $f(z) = \cos z$ en torno a $\frac{\pi}{2} = z_0$

Sabemos: $\cos\omega = 1 - \dfrac{\omega^2}{2!} + \dfrac{\omega^4}{4!} - \dfrac{\omega^6}{6!} + \dots = \sum_0^{\infty} \dfrac{(-1)^k}{(2k)!}\omega^{2k}$ para todo $\omega \in \mathbb{C}$

$\operatorname{sen}\omega = \omega - \dfrac{\omega^3}{3!} + \dfrac{\omega^5}{5!} - \dfrac{\omega^7}{7!} + \dots = \sum_0^{\infty} \dfrac{(-1)^k}{(2k+1)!}\omega^{2k+1}$

$$\cos z = -\operatorname{sen}(z - \pi/2) = -\left((z-\tfrac{\pi}{2}) - \frac{(z-\pi/2)^3}{3!} + \frac{(z-\pi/2)^5}{5!} - \dots\right) = \sum_0^{\infty} -\frac{(-1)^k (z-\frac{\pi}{2})^{2k+1}}{(2k+1)!}$$

$$\cos z = \sum_0^{\infty} \frac{(-1)^{k+1}(z-\pi/2)^{2k+1}}{(2k+1)!} \quad \text{para todo } z.$$

Coef: $a_j = \begin{cases} 0 & \text{si } j \text{ es par} \\ \dfrac{(-1)^{k+1}}{(2k+1)!} & \text{si } j \text{ es impar, } j = 2k+1 \end{cases}$

Radio conv: $R = \infty$

Otra forma:

$$a_0 = f(\pi/2) = \cos(\pi/2) = 0$$

$$a_1 = f'(\pi/2) = -\operatorname{sen}(\pi/2) = -1$$

$$a_2 = \frac{f''(\pi/2)}{2!} = \frac{-\cos(\pi/2)}{2!} = 0$$

$$\cos z = -1\left(z-\frac{\pi}{2}\right) + \frac{1}{3!}\left(z-\frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(z-\frac{\pi}{2}\right)^5 + \cdots$$

$$a_3 = \frac{f'''(\pi/2)}{3!} = \frac{\operatorname{sen}(\pi/2)}{3!} = \frac{1}{3!}$$

$$a_4 = \frac{f^{(4)}(\pi/2)}{4!} = \frac{\cos(\pi/2)}{4!} = 0$$

$$a_5 = \frac{f^{(5)}(\pi/2)}{5!} = \frac{-\operatorname{sen}(\pi/2)}{5!} = -\frac{1}{5!}$$

③ $f(z) = \frac{1}{z}$ en torno a $z_0 = i$

$$\frac{1}{z} = \frac{1}{(z-i)+i} = \frac{1}{i} \cdot \frac{1}{1+\frac{(z-i)}{i}}$$

Sabemos: $\frac{1}{1-\omega} = 1 + \omega + \omega^2 + \omega^3 + \cdots = \sum_{k=0}^{\infty} \omega^k$ si $|\omega| < 1$

$\left|-\frac{(z-i)}{i}\right| < 1$

$$\Rightarrow \frac{1}{z} = \frac{1}{i} \cdot \frac{1}{1-\left(-\frac{(z-i)}{i}\right)} = \frac{1}{i}\left[1 + \left(-\frac{(z-i)}{i}\right) + \left(-\frac{(z-i)}{i}\right)^2 + \left(-\frac{(z-i)}{i}\right)^3 + \cdots\right]$$

$$= \frac{1}{i}\sum_{k=0}^{\infty}\left(-\frac{(z-i)}{i}\right)^k = \sum_{k=0}^{\infty} \frac{-i(-1)^k}{i^k}(z-i)^k = -i + (z-i) - \frac{1}{i}(z-i)^2 + \frac{1}{i^2}(z-i)^3 + \cdots$$

si $\left|\frac{(z-i)}{i}\right| < 1 \iff |z-i| < 1$

$$a_k = \frac{-i(-1)^k}{i^k} = \frac{(-1)^{k+1}}{i^{k-1}}$$

$R = 1.$

④ $\boxed{z_0 = 0}$

$|z|<1$

$$\frac{1}{(1-z)^2} = \left(\frac{1}{1-z}\right)' = \left(1+z+z^2+z^3+\dots\right)' = \left(\sum_{k=0}^{\infty} z^k\right)' =$$

$$= \left(1+2z+3z^2+4z^3+\dots\right) = \sum_{k=1}^{\infty} k\cdot z^{k-1}$$

$$\frac{1}{(1-z)^2} = 1+2z+3z^2+4z^3+\dots = \sum_{0}^{\infty} k z^{k-1} \quad \text{si } |z|<1$$

$\left|-\frac{2}{3}z\right|<1$

$$f(z) = \frac{1}{(3+2z)^2} = \frac{1}{9}\frac{1}{\left(1+\frac{2z}{3}\right)^2} = \frac{-1}{9}\cdot\left(\frac{-3}{2}\right)\left(\frac{1}{1+\frac{2}{3}z}\right)' = -\frac{1}{6}\left(\frac{1}{1+\frac{2}{3}z}\right)' =$$

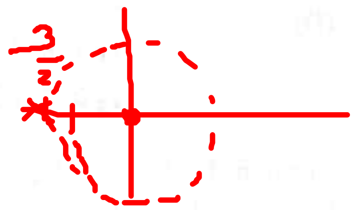
$$= -\frac{1}{6}\left(1-\frac{2}{3}z+\left(\frac{2}{3}z\right)^2+\left(-\frac{2}{3}z\right)^3+\left(-\frac{2}{3}z\right)^4+\dots\right)' = \left(\sum_{k=0}^{\infty} -\frac{1}{6}\left(-\frac{2}{3}\right)^k\cdot z^k\right)'$$

$$= -\frac{1}{6}\left(-\frac{2}{3}+\left(-\frac{2}{3}\right)^2\cdot 2z+\left(-\frac{2}{3}\right)^3\cdot 3z^2+\left(-\frac{2}{3}\right)^4\cdot 4z^3+\dots\right) = \sum_{k=1}^{\infty} -\frac{1}{6}\left(-\frac{2}{3}\right)^k k\, z^{k-1}$$

$$= \sum_{j=0}^{\infty} -\frac{1}{6}\left(-\frac{2}{3}\right)^{j+1}(j+1)\cdot z^j \qquad \text{si } \left|-\frac{2}{3}z\right|<1 \iff |z|<\frac{3}{2}$$

$$a_j = -\frac{1}{6}\left(-\frac{2}{3}\right)^{j+1}(j+1)$$

$R = 3/2$



$$f(z) = \frac{12-z}{2z^2-6z-20} = \frac{1}{2(z-5)} - \frac{1}{z+2}$$

$$\frac{1}{2(z-5)} = -\frac{1}{10}\cdot\frac{1}{1-\frac{z}{5}} = -\frac{1}{10}\left(1+\frac{z}{5}+\left(\frac{z}{5}\right)^2+\left(\frac{z}{5}\right)^3+\dots\right) = \sum_{k=0}^{\infty} -\frac{1}{10}\cdot\frac{1}{5^k} z^k$$

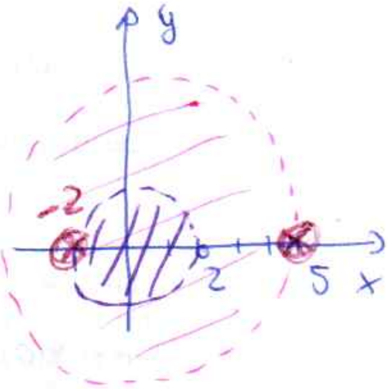
$\left|\frac{z}{5}\right|<1$ si $|z|<5$

$$\frac{1}{z+2} = \frac{1}{2}\cdot\frac{1}{1-\left(-\frac{z}{2}\right)} = \frac{1}{2}\left(1-\frac{z}{2}+\left(-\frac{z}{2}\right)^2+\left(-\frac{z}{2}\right)^3+\dots\right) = \sum_{k=0}^{\infty} \frac{1}{2}\frac{(-1)^k}{2^k} z^k \quad \text{si } |z|<2$$

$\left|-\frac{z}{2}\right|<1 \iff |z|<2$

$$f(z) = \frac{1}{2(z-5)} - \frac{1}{z+2} = \sum_{k=0}^{\infty} \overbrace{-\frac{1}{10}\cdot\frac{1}{5^k}\, z^k}^{|z|<5} - \sum \overbrace{\frac{1}{2}\left(\frac{-1}{2}\right)^k \cdot z^k}^{|z|<2}$$

$$= \sum_{k=0}^{\infty} \left(-\frac{1}{10}\frac{1}{5^k} - \frac{1}{2}\left(-\frac{1}{2}\right)^k \right) \cdot z^k =$$



$$a_k = -\frac{1}{10 . 5^k} - \frac{(-1)^k}{2^{k+1}} \qquad k = 0,1,2,\dots$$

$$R = \min\{5,2\} = 2 .$$

$$f^{(10)}(0) = a_{10} \cdot 10! = \left(-\frac{1}{10.5^{10}} - \frac{(-1)^{10}}{2^{11}} \right) 10!$$

(5) $f(z) = ch(z) = \cos(iz)$

$z_0 = 0$: $\quad f(z) = \sum_0^{\infty} (-1)^k \frac{(iz)^{2k}}{2k!} = 1 - \frac{i^2 z^2}{2!} + \frac{i^4 z^4}{4!} - \frac{i^6 z^6}{6!} - \dots$ para todo z

$$= 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \dots = \sum_0^{\infty} \frac{z^{2k}}{(2k)!}$$

$$a_j = \begin{cases} 0 & \text{si } j \text{ es impar} \\ \frac{1}{(2k)!} & \text{si } j \text{ es par, } j = 2k \end{cases} \qquad a_j = \begin{cases} 0 & \text{si } j \text{ impar} \\ \frac{1}{j!} & \text{si } j \text{ es par.} \end{cases}$$

$R = \infty$

$z_0 = \frac{\pi}{2} i \qquad f(z) = ch(z) = \cos(iz) = sen\left(iz + \pi/2\right) = sen\left(i\left(z - \frac{\pi}{2} i\right)\right)$

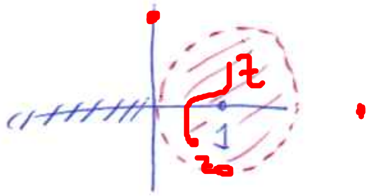
$$= i\left(z - \frac{\pi}{2} i\right) - \frac{\left[i\left(z - \frac{\pi}{2} i\right)\right]^3}{3!} + \frac{\left[i\left(z - \pi/2\, i\right)\right]^5}{5!} - \dots$$

$$= \sum_0^{\infty} (-1)^k i^{2k+1} \frac{(z - \pi/2\, i)^{2k+1}}{(2k+1)!} = \sum (-1)^k i \cdot i^{2k} \frac{(z - \pi/2\, i)^{2k+1}}{(2k+1)!} = \sum_0^{\infty} i \frac{(z - \frac{\pi}{2} i)^{2k+1}}{(2k+1)!}$$

$$a_j = \begin{cases} 0 & \text{si } j \text{ par} \\ (-1)^k \cdot i / (2k+1)! & \text{si } j = 2k+1 \end{cases}$$

$R = \infty$

⑥ $f(z) = \operatorname{Log}(z)$, centro: $z_0 = 1$



$$f'(z) = \frac{1}{z} \quad \text{si } z \in \mathbb{C} - \{z : \operatorname{Re} z \le 0, \operatorname{Im} z = 0\}$$

$$f'(z) = \frac{1}{1+z-1} \overset{|z-1|<1}{=} 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots = \sum_{k=0}^{\infty} (-1)^k (z-1)^k$$

$$\Rightarrow f(z) = \int_C f'(z)\,dz = \int_C \sum_0^{\infty} (-1)^k (z-1)^k\,dz \overset{\text{c.u.}}{=} \sum_{k=0}^{\infty} \int_C (-1)^k (z-1)^k\,dz$$

$$= \sum_{k=0}^{\infty} (-1)^k \cdot \frac{(z-1)^{k+1}}{k+1} + c = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots + c$$

$$f(1) = \operatorname{Log}(1) = 0 = c \quad \Rightarrow c = 0.$$

$$f(z) = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots = \sum_{k=0}^{\infty} (-1)^k \frac{(z-1)^{k+1}}{k+1} \qquad |z-1|<1$$

$$= \sum_{j=1}^{\infty} (-1)^{j-1} \frac{(z-1)^j}{j}$$

$$a_j = \begin{cases} 0 & j = 0 \\ \dfrac{(-1)^{j-1}}{j} & j \ge 1 \end{cases}$$

⑦ $f(z) = \dfrac{e^z}{1+z}$ $\qquad z_0 = 0$

$$f(z) = \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots\right)\left(1 - z + z^2 - z^3 + z^4 - \cdots\right)$$

$$= 1 + (1-1)z + \left(1 - 1 + \frac{1}{2!}\right)z^2 + \left(-1 + 1 - \frac{1}{2!} + \frac{1}{3!}\right)z^3 + \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)z^4 + \cdots$$

$$= 1 + \frac{1}{2!}z^2 + \left(\frac{-1}{2!} + \frac{1}{3!}\right)z^3 + \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)z^4 + \cdots$$

$$f^{(4)}(0) = a_4 \cdot 4! = \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) \cdot 4! = 3 \cdot 4 - 4 + 1 = 9$$